CONSTRUCTION OF MODELS OF CONTINUOUS MEDIA BY MEANS OF THE VARIATIONAL PRINCIPLE

(POSTROENIE MODELEI SPLOSHNYKH SRED PRI POMOSHCHI VARIATSIONNOGO PRINTSIPA)

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V.L. BERDICHEVSKII
(Moscow)

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We present a formal method of construction of models of continuous media within the framework of the General Theory of Relativity. Starting from the variational principle formulated in [1], we obtain the equations of state, together with a closed system of differential equations describing the continuous medium, the determining parameters of which include the first and second derivatives of the laws of motion and of the field functions. We note that a continuous medium is characterised by three, generally distinct, tensors of the energy impulse. We consider a series of models; we also show how it is possible to arrive from the derived formulas to the corresponding expressions in the Newtonian mechanics.

1. General procedure in the construction of models of continuous media. Let us consider the construction of models of continuous media by means of the variational principle expressed in the form [1]

$$\delta \int_V \Lambda \, d\tau + \delta W + \delta W^* = 0 \tag{1.1}$$

Here V is an arbitrary volume of a four-dimensional Riemann manifold of states G, Λ is a Lagrangian which we shall regard as a four-dimensional scalar, and δW^* is the given functional. The quantity δW is expressed as an integral along the boundary of the volume V, the boundary being a three-dimensional region Σ, of a linear combination of the variations of the defining parameters and their derivatives, and it is fully determinable if functions Λ and δW^* are known.

The model of the continuous medium will be defined if the Lagrangian Λ and δW^* are known. For instance, for a model of an ideal fluid with reversible processes in the Special Theory of Relativity we have, as will be shown later, $\Lambda = -\rho U(\rho S)$, where ρ is the density of the rest mass of the fluid, U is the internal energy and S is the entropy.

Further we shall assume that the arguments of Λ include the following quantities:

$$g_{ij}, \quad \frac{\partial g_{ij}}{\partial x^k}, \quad \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}, \quad x^i{}_p = \frac{\partial x^i}{\partial \xi^p}, \quad \nabla_k x^i{}_p, \quad \mu^A, \quad \nabla_j \mu^A, \quad \nabla_i \nabla_j \mu^A, \qquad K^C, \quad \nabla_j K^C$$

Here*, g_{ij} are the components of the metric tensor of the space-time manifold G in the reference system of the observer, the coordinates of which are denoted by x^i and ξ^p are the coordinates in the associated frame of reference, with the coordinate lines ξ^4 coinciding with the world lines of the particles of the medium the Lagrangian coordinates of which are ξ^1, ξ^2 and ξ^3. Functions $x^i(\xi^p)$, defined in some region of the manifold G, fully describe the motion of the medium**. When the associated frame of reference is assumed fixed, the complex $x^i{}_p = \partial x^i / \partial \xi^p$ forms a vector of index i in the reference system of the observer. All the covariant derivatives, corresponding to the symbol ∇_k, are taken in the reference system x^i, i.e.

$$\nabla_k x^i{}_p = \partial x^i{}_p / \partial x^k + \Gamma_{ks}{}^i x^s{}_p$$

The variables μ^A describe either the state of the medium (temperature, magnetisation, curvature and rotation of a set of fault-free states in a continuous theory of dislocations, etc.), or the fields present (for instance, electromagnetic). The constant quantities K^C describe the properties of the medium (anisotropy, dielectric permeability, metric tensor of the set of initial states, etc.).

In the works of Sedov [1 and 2] a detailed study was made of the case when the Lagrangian depends upon $x^i(\xi^p)$, and $\mu^A(x^i)$ and their first derivatives. The inclusion of the gradients of x^i_p and $\nabla_j \mu^A$ into the arguments of Λ makes it possible to take into account some new effects such as the internal moment of momentum of the continuous medium.

In Equation (1.1), μ^A is subjected to variations together with the metric of the manifold G

$$\partial g_{ij} = g_{ij}'(x) - g_{ij}(x)$$

and the trajectories of the particles of the medium

$$x'^i(\xi^p) = x^i(\xi^p) + \varphi^i(\xi^p, \varepsilon), \qquad \varphi^i(\xi^p, \varepsilon) = \sum_{n=1}^{\infty} \varphi^i{}_{(n)}(\xi^p)\, \varepsilon^n.$$

As usual, vector δx^i is the principal, linear in ε, part of the difference $x'^i - x^i$. When μ^A, assumed to be attached to the points of the medium are varied, it is necessary

* All lower case italics $i, j, k, p, q, \ldots$, range through the values 1, 2, 3 and 4, while the capital indices A, B and C, can correspond to one or more tensorial indices (an analogous convention could also be constructed for the spinor indices A, B and C). Lower case greek letters $\alpha, \beta, \gamma, \ldots$ range through the values 1, 2 and 3 and correspond to the spatial coordinates.

** It should be noted that the present analysis is confined to the case of homogeneous media where Λ does not explicitly depend upon x^i or ξ^p.

to distinguish between the full variation $\delta\mu^A$ and the variation of μ^A at the point $x - \delta\mu^A$. By definition*,

$$\delta\mu^A \equiv [\mu'^A (x' (\xi))]^= - \mu^A (x (\xi)) = \partial\mu^A + \delta x^i \nabla_i \mu^A$$

Here $[\mu'^A (x'(\xi))]^=$ denotes the result of a parallel translation of $\mu'^A (x)$ from the point x' to x, i.e.

$$[\mu'^A (x')]^= = \mu'^A (x) + \delta x^i \nabla_i \mu^A$$

It is not difficult to verify the following expressions:

$$\delta x^i_p = [x'^i_p (x')]^= - x^i_p (x) = x'^i_p (x' (\xi)) + \Gamma_{sl}{}^i x^s{}_p \delta x^l - x_p{}^i (x (\xi)) =$$

$$= \frac{\partial}{\partial \xi^p} (x'^i (\xi) - x^i (\xi)) + \Gamma_{sl}{}^i x^s{}_p \delta x^l = \frac{\partial \delta x^i}{\partial x^s} x^s{}_p + \Gamma_{sl}{}^i \delta x^l x^s{}_p = x^s{}_p \nabla_s \delta x^i =$$

$$= \partial x^i{}_p + \delta x^l \nabla_l x^i{}_p$$

$$\delta \nabla_k x^i_p = \partial \nabla_k x^i{}_p + \delta x^l \nabla_l \nabla_k x^i{}_p = \nabla_k \partial x^i{}_p + \frac{\partial \nabla_k x^i{}_p}{\partial \Gamma_{lm}{}^n} \partial \Gamma_{lm}{}^n + \delta x^l \nabla_l \nabla_k x^i{}_p =$$

$$= \nabla_k (x^j{}_p \nabla_j \delta x^i - \delta x^l \nabla_l x^i{}_p) + \frac{\partial \nabla_k x^i{}_p}{\partial \Gamma_{lm}{}^n} \partial \Gamma_{lm}{}^n + \delta x^l \nabla_l \nabla_k x^i{}_p =$$

$$= \delta x^l (\nabla_l \nabla_k x^i{}_p - \nabla_k \nabla_l x^i{}_p) + \nabla_j \delta x^l (\delta_l{}^i \nabla_k x^j{}_p - \delta_k{}^j \nabla_l x^i{}_p) +$$

$$+ x^j{}_p \nabla_k \nabla_j \delta x^i + \frac{\partial \nabla_k x^i{}_p}{\partial \Gamma_{lm}{}^n} \partial \Gamma_{lm}{}^n$$

$$\delta \nabla_j \mu^A = \nabla_j \partial \mu^A + \frac{\partial \nabla_j \mu^A}{\partial \Gamma_{lm}{}^n} \partial \Gamma_{lm}{}^n + \delta x^i \nabla_i \nabla_j \mu^A =$$

$$= \nabla_j \delta \mu^A + \delta x^i (\nabla_i \nabla_j \mu^A - \nabla_j \nabla_i \mu^A) - (\nabla_j \delta x^i) \nabla_i \mu^A + \frac{\partial \nabla_j \mu^A}{\partial \Gamma_{lm}{}^n} \partial \Gamma_{lm}{}^n$$

$$\delta \nabla_s \nabla_j \mu^A = \nabla_s \partial \mu^A{}_j + \frac{\partial \nabla_s \mu^A{}_j}{\partial \Gamma_{lm}{}^n} \partial \Gamma_{lm}{}^n + \delta x^i \nabla_i \nabla_s \nabla_j \mu^A =$$

$$= \nabla_s \left(\nabla_j \partial \mu^A + \frac{\partial \nabla_j \mu^A}{\partial \Gamma_{lm}{}^n} \partial \Gamma_{lm}{}^n \right) + \frac{\partial \nabla_s \mu^A{}_j}{\partial \Gamma_{lm}{}^n} \partial \Gamma_{lm}{}^n + \delta x^i \nabla_i \nabla_s \nabla_j \mu^A = \nabla_s \nabla_j \delta \mu^A +$$

$$+ \delta x^i (\nabla_i \nabla_s \nabla_j \mu^A - \nabla_s \nabla_j \nabla_i \mu^A) - \nabla_k \delta x^i (\delta_j{}^k \nabla_s \nabla_i \mu^A + \delta_s{}^k \nabla_j \nabla_i \mu^A) -$$

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* Usually $\delta\mu^A$ is defined thus: $\delta_1\mu^A = \mu'^A (x') - \mu^A (x)$, where $\delta_1\mu^A$ is no longer a tensor. This definition, however, has some disadvantages; thus, for instance, the quantities K^C ($\delta K^C = 0$) which were not variable in the first definition, must be regarded as variable in the second. The use of variations $\delta\mu^A$ is possible by virtue of the equality

$$\delta\Lambda = \Lambda (\mu'^A(x')) - \Lambda (\mu^A (x)) = [\Lambda (\mu'^A(x'))]^= - \Lambda (\mu^A(x)) = \Lambda ([\mu'^A(x')]^=) - \Lambda (\mu^A(x))$$

$$-(\nabla_s\nabla_j\delta x^i)\cdot\nabla_i\mu^A + \partial\Gamma_{lm}{}^n\left(\frac{\partial\nabla_s\mu^A{}_j}{\partial\Gamma_{lm}{}^n} + \nabla_s\frac{\partial\nabla_j\mu^A}{\partial\Gamma_{lm}{}^n}\right) + \frac{\partial\nabla_j\mu^A}{\partial\Gamma_{lm}{}^n}\nabla_s\partial\Gamma_{lm}{}^n$$

Here, μ^A_j denotes tensor $\nabla_j\mu^A$, and the symbol $\partial\nabla_s\mu^A{}_j / \partial\Gamma_{lm}{}^n$ shows that the derivative with respect to $\Gamma_{lm}{}^n$ should be taken of the first covariant derivative of tensor μ^A_j, and not of the second covariant derivative of tensor μ^A. The variation of the Christoffel symbol $\partial\Gamma_{lm}{}^n$, appearing in the above formulas is a tensor of third rank which can be expressed in terms of the covariant derivatives of variations of the metric

$$\partial\Gamma_{lm}{}^n = B_{lm}{}^{nkij}\nabla_k\partial g_{ij}$$

$$B_{lm}{}^{nkij} = \frac{1}{4}\left[-g^{nk}(\delta_l{}^i\delta_m{}^j + \delta_m{}^i\delta_l{}^j) + g^{nj}(\delta_l{}^i\delta_m{}^k + \delta_m{}^i\delta_l{}^k) + g^{ni}(\delta_l{}^j\delta_m{}^k + \delta_m{}^j\delta_l{}^k)\right]$$

The variations of the metric can be conveniently expressed by [2]

$$\partial g_{ij} = \partial^* g_{ij} + \nabla_i\eta_j + \nabla_i\eta_j$$

where η_i is an arbitrary small covector. In the special theory of relativity (where the tensor of the curvature of the space-time manifold $R_{ijk}{}^l = 0$) ,by definition $\partial^* g_{ij} = 0$, and it is easily seen that the changes of the metric $\nabla_i\eta_j + \nabla_j\eta_i$, leave the tensor of curvature $R_{ijk}{}^l$ equal to zero, i.e. they remain within the Euclidian space.

Using the above variational formulas and the equality

$$V^i\Delta_i x\varrho + V\varrho = V\varrho$$

one can write* :

$$\delta\int_V \Lambda\, d\tau = \int_V\left\{-\frac{1}{2}\Theta^{ij}\partial^* g_{ij} + \eta_i\nabla_j\Theta^{ij} + M_A\delta\mu^A + X_i\delta x^i\right\}d\tau -$$

$$-\int_\Sigma\{\Theta^{il}\eta_i + S^{ijl}\partial g_{ij} + S^{ijkl}\nabla_k\partial g_{ij} + P_A{}^l\delta\mu^A + P_A{}^{il}\nabla_i\delta\mu^A +$$

$$+ P_i{}^l\delta x^i + P_i{}^{jl}\nabla_j\delta x^i\}\, n_l\, d\sigma$$

Here, η_l is the vector of the normal to Σ, and other symbols have the following meaning** :

$$-\frac{1}{2}\Theta^{ij} = \frac{1}{\sqrt{-g}}\left(\frac{\partial\Lambda\sqrt{-g}}{\partial g_{ij}} - \frac{\partial}{\partial x^k}\frac{\partial\Lambda\sqrt{-g}}{\partial(\partial g_{ij}/\partial x^k)} + \frac{\partial^2}{\partial x^k\partial x^l}\frac{\partial\Lambda\sqrt{-g}}{\partial(\partial^2 g_{ij}/\partial x^k\partial x^l)}\right) + \quad (1.2)$$

* It should be borne in mind that the coefficients of the variation δx^i appearing in the surface integral, will be different, depending on whether the variations in the corresponding terms are $\delta\mu^A$ or $\delta\mu^A$. Because of this, it is possible to have different definitions for tensor P^J_i.

** In the derivation of Equations (1.2) to (1.10) the fact that both, Λ and the volume V of the space-time manifold G are subjected to variations, was taken into account. For the purpose of variations of an element of volume, one can use the formula

$$\delta(d\tau) = (\partial\sqrt{-g}/\sqrt{-g} + \nabla_i\delta x^i)\, d\tau, \quad g = \det\|g_{ij}\|$$

$$+ \nabla_l \psi^{ijl} + \frac{1}{2} \frac{\partial \Lambda}{\partial \nabla_s \nabla_p \mu^A} \frac{\partial \nabla_p \mu^A}{\partial \Gamma_{lm}{}^n} B_{lm}{}^{nkqr} (R_{skq}{}^i \delta_r{}^j + R_{skr}{}^j \delta_q{}^i)$$

$$M_A = \frac{\partial \Lambda}{\partial \mu^A} - \nabla_j \frac{\partial \Lambda}{\partial \nabla_j \mu^A} + \nabla_j \nabla_i \frac{\partial \Lambda}{\partial \nabla_i \nabla_j \mu^A} \tag{1.3}$$

$$X_i = \nabla_j (P_i{}^j + \Lambda \delta_i{}^j) - \frac{\partial \Lambda}{\partial x^j{}_s} \nabla_i x^j{}_s - \frac{\partial \Lambda}{\partial \nabla_j x^p{}_s} \nabla_j \nabla_i x^p{}_s - \frac{\partial \Lambda}{\partial \mu^A} \nabla_i \mu^A - \tag{1.4}$$
$$- \frac{\partial \Lambda}{\partial \nabla_j \mu^A} \nabla_j \nabla_i \mu^A - \frac{\partial \Lambda}{\partial \nabla_s \nabla_j \mu^A} \nabla_s \nabla_j \nabla_i \mu^A - \frac{\partial \Lambda}{\partial K^C} \nabla_i K^C - \frac{\partial \Lambda}{\partial \nabla_j K^C} \nabla_i \nabla_j K^C +$$
$$+ \frac{1}{2} \frac{\partial \Lambda}{\partial \nabla_k x^j{}_p} R_{kli}{}^j x^l{}_p - \frac{1}{2} \frac{\partial \Lambda}{\partial \nabla_s \nabla_j \mu^A} R_{sji}{}^l \nabla_l \mu^A$$

$$S^{ijl} = \Psi^{ijl} - \frac{\partial \Lambda}{\partial (\partial g_{ij} / \partial x^l)} + \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^k} \frac{\partial \Lambda \sqrt{-g}}{\partial (\partial^2 g_{ij} / \partial x^k \partial x^l)} - \tag{1.5}$$
$$- \frac{\partial \Lambda}{\partial (\partial^2 g_{pq} / \partial x^k \partial x^l)} (\Gamma_{kp}{}^i \delta_q{}^j + \Gamma_{kq}{}^j \delta_p{}^i)$$

$$\Psi^{ijl} = - \left[\frac{\partial \Lambda}{\partial \nabla_q x^s{}_p} \frac{\partial \nabla_q x^s{}_p}{\partial \Gamma_{km}{}^n} + \frac{\partial \Lambda}{\partial \nabla_p \mu^A} \frac{\partial \nabla_p \mu^A}{\partial \Gamma_{km}{}^n} + \frac{\partial \Lambda}{\partial \nabla_p \nabla_q \mu^A} \left(\frac{\partial \nabla_p \mu^A{}_q}{\partial \Gamma_{km}{}^n} + \nabla_p \frac{\partial \nabla_q \mu^A}{\partial \Gamma_{km}{}^n} \right) \right] \times$$
$$\times B_{km}{}^{nlij} + \nabla_s \left[\frac{1}{2} \frac{\partial \Lambda}{\partial \nabla_s \nabla_p \mu^A} \frac{\partial \nabla_p \mu^A}{\partial \Gamma_{km}{}^n} B_{km}{}^{nlij} + (j \leftrightarrow s) \right] \tag{1.6}$$

$$S^{ijkl} = - \frac{\partial \Lambda}{\partial (\partial^2 g_{ij} / \partial x^k \partial x^l)} - \left[\frac{1}{2} \frac{\partial \Lambda}{\partial \nabla_k \nabla_p \mu^A} \frac{\partial \nabla_p \mu^A}{\partial \Gamma_{qm}{}^n} B_{qm}{}^{nlij} + (k \leftrightarrow l) \right] \tag{1.7}$$

$$P_A{}^j = - \frac{\partial \Lambda}{\partial \nabla_j \mu^A} - \nabla_l P_A{}^{jl}, \qquad P_A{}^{jl} = - \frac{1}{2} \left[\frac{\partial \Lambda}{\partial \nabla_l \nabla_j \mu^A} + (j \leftrightarrow l) \right] \tag{1.8}$$

$$P_i{}^j = \nabla_i \mu^A \frac{\partial \Lambda}{\partial \nabla_j \mu^A} + \frac{\partial \Lambda}{\partial \nabla_s \nabla_k \mu^A} (\delta_k{}^j \nabla_s \nabla_i \mu^A + \delta_s{}^j \nabla_k \nabla_i \mu^A) - x^j{}_s \frac{\partial \Lambda}{\partial x^i{}_s} -$$
$$- \frac{\partial \Lambda}{\partial \nabla_k x^l{}_s} (\delta_i{}^l \nabla_k x^j{}_s - \delta_l{}^j \nabla_i x^l{}_s) - \nabla_l P_i{}^{jl} - \Lambda \delta_i{}^j \tag{1.9}$$

$$P_i{}^{jl} = \frac{1}{2} \left[\nabla_i \mu^A \frac{\partial \Lambda}{\partial \nabla_j \nabla_l \mu^A} + (l \leftrightarrow j) \right] - \frac{1}{2} \left[x^l{}_s \frac{\partial \Lambda}{\partial \nabla_j x^i{}_s} + (l \leftrightarrow j) \right] \tag{1.10}$$

Symbol $(k \to l)$ in Equations (1.2) to (1.10) indicates, that the term followed by it should be supplemented with another term, identical to it, but with the suffices k and l interchanged. Tensor $R_{ijk}{}^l$ appearing in (1.2) and (1.4) is the tensor of curvature of the space-time manifold.

It is easily seen, that the tensors $P_i{}^j$, $P_i{}^{jl}$, etc. thus defined together with V depend, in general, on the choice of the associated reference system, since the latter affects the tensor $\nabla_k x^i{}_r$. It will be assumed later that the function Λ and the components of the tensors $P_i{}^j$, $P_i{}^{jl}$, etc it defines given in the observers references system, do not depend on the choice of the associated frame of reference. This assumption will be valid when, for instance, $\nabla_k x^i{}_p$ are included in Λ in the following manner: first x^i_p is contracted in p with some tensor Q^{Bp}, and then derivative ∇_k is taken. Such approach was adopted in the majority of earlier models.

We shall further consider the processes when

$$\delta W^* = 0$$

The definition of δW^* is an entirely separate problem which will not be considered in this paper.

Assuming that the variations of the variables and their derivatives are equal to zero on Σ and that δW is given as an integral with respect to Σ of linear combinations of variations of the variables and their derivatives, we obtain from (1.1)

$$\Theta^{ij} = 0, \quad \text{if} \quad \partial^* g_{ij} \neq 0, \ \nabla_j \Theta^{ij} = 0, \quad M_A = 0, \quad X_i = 0 \qquad (1.11)$$

In addition, we have from equation (1.1) for the variations differing from zero on Σ

$$\delta W = \int_\Sigma \{\Theta^{il}\eta_i + S^{ijl}\partial g_{ij} + S^{ijkl}\nabla_k \partial g_{ij} + P_A{}^l \delta\mu^A + $$
$$+ P_A{}^{jl}\nabla_j \delta\mu^A + P_i{}^l \delta x^i + P_i{}^{jl}\nabla_j \delta x^i\} n_l \, d\sigma$$

Equation (1.1) is a generalised form of the principle of conservation of energy and it contains terms describing the work done by virtual changes (variations) in the metric, field functions and trajectories of the particles of the medium. Let us examine the individual terms entering δW to see what are energy contributions and states. The tensor $P_i{}^j$ performs work during changes in the world lines of the particles of the medium at the boundary of region V; i.e. it express to the four-dimensional generalisation of a three-dimensional stress tensor a tensor of the energy impulse. Apart from $P_i{}^j$, some work is done during changes in the trajectory by some double forces $P_i{}^{jl}$ which will be defined more accurately later. The terms of δW, containing tensors Θ^{ij}, S^{ijl}, and S^{ijkl}, define the work associated with changes in the metric, but the physical sense of tensors $P_A{}^j$ and $P_A{}^{jl}$ depends on the nature of the quantities μ^A.

The laws of conservation of energy involving $P_i{}^j$, $P_i{}^{jl}$, $P_A{}^j$, and $P_A{}^{jl}$, can be formulated by means of Netter's theorem. We stipulate that the integral of the Lagrangian Λ is invariant with respect to the group, containing r parameters, of motions of the Riemanian manifold G (i.e. group which does not alter the metric of G), over the four-dimensional region V. Let the action of this group on x^i and μ^A produce the following variations

$$\delta x^i = X_a{}^i \delta\omega^a, \qquad \delta\mu^A = M_a^A \delta\omega^a$$

Here $\delta\omega^a$ denotes the parameters of the group, and it is assumed that $\nabla_k \delta\omega^a = 0$).

The corresponding variation of the integral of Λ over the region V is

$$\delta \int_V \Lambda \, d\tau = \int_\Sigma F_a{}^l \delta\omega^a n_l d\sigma$$

where

$$F_a{}^l = P_i{}^l X_a{}^i + P_i{}^{jl}\nabla_j X_a{}^i + P_A{}^l M_a^A + P_A^{jl}\nabla_j M_a^A \qquad (1.12)$$

From the condition of invariance of the integral of Λ taken over the region V, we obtain

$$\nabla_l F_a{}^l = 0 \qquad (a = 1,2, \ldots r) \tag{1.13}$$

In particular, within the framework of the Special Theory of Relativity in an inertial rectilinear reference system for the Lorentz group, whose parameters are the components of the antisymmetric tensor $\delta\omega_{ij}$, which are

$$\delta\mu^A = M^{Aij}\delta\omega_{ij}$$

Equation (1.13) assumes the form

$$\underbrace{\partial M^{ijk}}_{k} = P^{ij} - P^{ji}, \qquad M^{ijk} = P^{jik} - P^{ijk} + P_A{}^k M^{Aij} + P_A{}^{kl}\frac{\partial M^{Aij}}{\partial x^l} \tag{1.14}$$

In order to impart physical sense to the components of the tensor M^{ijk}, let us compare Equation (1.14) with a three-dimensional non-relativistic equation of the equilibrium of the internal moment of momentum expressed in the divergence form:

$$\frac{\partial}{\partial t}\rho m^{\alpha\beta} + \frac{\partial}{\partial x^\gamma}(\rho m^{\alpha\beta}v^\gamma - Q^{\alpha\beta\gamma}) = p^{\beta\alpha} - p^{\alpha\beta} + \rho h^{\alpha\beta} \tag{1.15}$$

Here $m^{\alpha\beta}$ denotes a tensor of the internal moment of momentum, v^γ is three-dimensional velocity, $Q^{\alpha\beta\gamma}$ is the surface internal momentum, $p^{\alpha\beta}$ is the stress tensor, $h^{\alpha\beta}$ is the internal mass momentum. We shall assume in the following that $h^{\alpha\beta} = 0$.

Since Equations (1.14) and (1.15) can be assumed to represent identical physical phenomena (when $h^{\alpha\beta} = 0$), we can have

$$\frac{1}{c}M^{\alpha\beta 4} = \rho m^{\alpha\beta}, \qquad M^{\alpha\beta\gamma} = \rho m^{\alpha\beta}v^\gamma - Q^{\alpha\beta\gamma}$$

where c is the velocity of light.

Thus, the 'double forces' $P_i{}^{jl}$ are connected with the presence of the internal angular momentum. They arise due to the fact that in the present formulation the argument of Λ include gradients of the quantities x^i_p and $\nabla_j\mu^A$; consequently, the models of continuous media which contain amongst their defining parameters $\nabla_k x^i_p$ or the second derivatives of the field functions exhibit, in general, an internal angular momentum.

Let us now examine equations (1.2)-(1.10) more closely on the basis of the General Theory of Relativity. One of the basic premises in the general theory is that [3]

$$\Lambda = R / 2\varkappa + \Lambda_m$$

where R is the scalar curvature of the manifold G, Λ_m is the Lagrangian of matter and $\varkappa$ = const (the arguments which follow can easily be applied also to the case when $\varkappa$ is a variable). Then, since $\partial^* g_{ij} \neq 0$, the first equation of (1.11) applies* and it assumes the Einsteinian form:

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* Let us note that in the Special Theory of Relativity $\partial^* g_{ij} = 0$ and the first equation of

(continued on the next page)

$$\Theta^{ij} = \frac{1}{\varkappa}\,(R^{ij} - \frac{1}{2}\,Rg^{ij}) + T^{ij} = 0$$

where R^{ij} is Ricci's tensor, and T^{ij} denotes

$$T^{ij} = -2\left\{\frac{1}{\sqrt{-g}}\Big(\frac{\partial\Lambda_m\sqrt{-g}}{\partial g_{ij}} - \frac{\partial}{\partial x^k}\frac{\partial\Lambda_m\sqrt{-g}}{\partial\,(\partial g_{ij}/\partial x^k)} + \frac{\partial^2}{\partial x^k\partial x^l}\frac{\partial\Lambda_m\sqrt{-g}}{\partial\,(\partial^2 g_{ij}/\partial x^k\partial x^l)}\Big) + \right.$$
$$\left. + \nabla_k\Psi^{ijk} + \frac{1}{2}\frac{\partial\Lambda}{\partial\nabla_s\nabla_p\mu^A}\frac{\partial\nabla_p\mu^A}{\partial\Gamma_{lm}{}^n}B_{lm}{}^{nkqr}\,(R_{skq}{}^i\delta_r{}^j + R_{skr}{}^j\delta_q{}^i)\right\} \tag{1.18}$$

By virtue of $\nabla_j\Theta^{ij} = 0$ and Bianchi's identities, tensor T^{ij} satisfies the equation

$$\nabla_j T^{ij} = 0$$

By virtue of this condition, it is assumed in the Special Theory of Relativity that T^{ij} is a tensor of energy impulse. As shown in Equation (1.16), the right-hand side of Einstein's equation denoted by T^{ij}, contains, in general, also a tensor of curvature of the time-space manifold $R_{ijk}{}^l$.

Let now assume tha the Lagrangian of matter Λ_m is a function of $\partial g_{ij}/\partial x^k$ and $\partial^2 g_{ij}/\partial x^k\partial x^l$ only by virtue of the covariant derivatives ∇_i, and $\nabla_i\nabla_j$. Then, the last equation of (1.11) can easily be put in the form:

$$\nabla_j P_{(m)i}{}^j + \frac{\partial\Lambda_m}{\partial\nabla_j x^l{}_s}x^k{}_s R_{ijk}{}^l + \frac{1}{2}\frac{\partial\Lambda_m}{\partial\nabla_j x^l{}_s}x^k{}_s R_{jki}{}^l - \frac{1}{2}\frac{\partial\Lambda_m}{\partial\nabla_j\nabla_k\mu^A}\nabla_l\mu^A R_{jki}{}^l +$$
$$+ \frac{\partial\Lambda_m}{\partial\nabla_j\mu^A}(\nabla_i\nabla_j\mu^A - \nabla_j\nabla_i\mu^A) + \frac{\partial\Lambda_m}{\partial\nabla_j\nabla_k\mu^A}(\nabla_i\nabla_j\nabla_k\mu^A - \nabla_j\nabla_k\nabla_i\mu^A) = 0 \tag{1.17}$$

where $P_{(m)i}^{\;j}$ denotes the tensor defined by Equation (1.9) but with Λ replaced by Λ_m. In the Special Theory of Relativity, Equation (1.17) assumes the usual form

$$\nabla_j P_{(m)i}^{\;j} = 0 \tag{1.18}$$

In the General Theory, the divergence of the tensor $P^j_{(m)i}$, which is a canonical tensor of the energy impulse, does no longer in general, reduce to zero, but is expressible as a linear function of the tensor of curvature $R_{ijk}^{\;l}$, in accordance with Equation (1.17).

(continued from previous page)
(1.11) is absent; Θ^{ij} is defined by the second equation of (1.11). The only change of the metric which is allowed by the Special Theory is $\partial^{**}g_{ij} = \nabla_i\eta_j + \nabla_j\eta_i$, which, broadly speaking, corresponds to the transition to a non-inertial reference system. At the same time, in the equation definining the balance of energy additional terms (1.1)

$$\int_\Sigma \Theta^{il}\eta_i n_l\,d\sigma, \qquad \int_\Sigma S^{ijl}\partial^{**}g_{ij}n_l\,d\sigma, \qquad \int_\Sigma S^{ijkl}\nabla_k\partial^{**}g_{ij}n_l\,d\sigma$$

appear.

Thus, in general case, there are three distinct tensors of the energy impulse: symmetrical T^{ij}, canonical $P_{(m)i}^{\ j}$ and $P_i^{\ j}$ where

$$P_i^{\ j} = P_{(m)\,i}^{\ \ \ j} - \frac{1}{2\varkappa} R\delta_i^{\ j}$$

In some models, $P_{(m)}^{ij}$ and T^{ij} are identical. This occurs, for instance, in the media in which $\Lambda_m = \Lambda_m\,(g^\wedge{}_{pq}, K^C)$, where $g^\wedge{}_{pq}$ is a metric tensor in the associated reference system. Such media include the ideal fluid, since its density ρ can be expressed in terms of $g^\wedge{}_{pq}$ (cf. Equations (3.3) and (2.3)).

2. Deformable elastic media. We shall next consider the models of continuous media in which the Lagrangian of matter depends on the quantities

$$g^\wedge{}_{pq},\quad u^\wedge{}_p,\quad \nabla^\wedge{}_r u^\wedge{}_p,\quad g^{\circ pq},\quad u^{\circ p},\quad \nabla^\wedge{}_r g^\circ{}_{pq},\quad \nabla^\wedge{}_r u^\circ{}_p \tag{2.1}$$

where $u^\wedge{}_r$ are the components of the 4-covector representing the velocity in the associated reference system, $g^{\circ pq}$ are the contravariant components of the metric tensor in the initial condition, $u^{\circ p}$ are the component of the 4-vector representing the velocity in the initial condition, and the symbol $\nabla^\wedge$ indicates that the covariant derivatives should be calculated in the associated reference system*.

For definiteness, we shall choose in the time-space manifold a metric which can, at every point, be brought to the form

$$g_{11} = g_{22} = g_{23} = -g_{44} = -1, \qquad g_{\alpha 4} = 0$$

with the differential of the arc length ds along the world line

$$ds^2 = g_{ij}\,dx^i dx^j = g^\wedge{}_{44} d\xi^{4^2}$$

being a real quantity $(g^\wedge{}_{44} > 0)$, and the element of the spatial distance a purely imaginary quantity. The radius of the 4-vector of the velocity is, in this convention, positive and equal to + 1. Components of the vector u^j in the observer's reference system x^i are given by

$$u^j = \frac{dx^j}{ds} = \left(g_{lm}\frac{\partial x^l}{\partial \xi^4}\frac{\partial x^m}{\partial \xi^4}\right)^{-1/2}\frac{\partial x^j}{\partial \xi^4} = \frac{x^j{}_4}{\sqrt{g^\wedge{}_{44}}} \tag{2.2}$$

By definition we have, in the associated coordinate system

$$u^{\wedge p} = \frac{\delta_4{}^p}{\sqrt{g^\wedge{}_{44}}}, \qquad u^\wedge{}_p = g^\wedge{}_{pq} u^{\wedge q} = \frac{g^\wedge{}_{p4}}{\sqrt{g^\wedge{}_{44}}} \tag{2.3}$$

Also by definition, we shall treat the space of initial conditions as a Riemann space with the metric defined by

$$g^\circ{}_{pq}(\xi^\alpha, \xi^4) = g^\wedge{}_{pq}(\xi^\alpha, \xi_0{}^4), \qquad \xi_0{}^4 = \text{const}, \qquad \nabla^\wedge{}_4 g^\circ{}_{pq} \neq 0$$

* From now on, the symbol $^\wedge$ will be used to denote that the relevant quantity belongs to the associated reference system. Tensors pertaining to the observer's reference system will be shown without this symbol, as before.

4-vector of the velocity in the initial conditions will be defined by

$$u^\circ_p(\xi^\alpha, \xi^4) = u^\wedge_p(\xi^\alpha, \xi_0^4), \qquad \xi_0^4 = \text{const}, \qquad \nabla^\wedge_4 u^\circ_p \neq 0$$

By definition, tensors $g^{\circ pq}$ and $u^{\circ p}$ will denote

$$g^{\circ pq} = \frac{1}{g^\circ}\frac{\partial g^\circ}{\partial g^\circ_{pq}}, \qquad g^\circ = \det\|g^\circ_{pq}\|, \qquad u^{\circ p} = g^{\circ pq}u^\circ_q$$

It can be easily seen that the tensor fields defined in this way depend upon the specific associated system in which they were constructed. This, however, will not be the case if the reference systems are confined only to those which allow transition from one to the other by means of the transformations

$$\eta^\alpha = \eta^\alpha(\xi^\beta), \qquad \eta^4 = \xi^4 + \varphi(\xi^\alpha) \tag{2.4}$$

In the following, we shall allow only such associated reference systems for which transition formulas will be of the type (2.4). This restriction still leaves us a wide choice of the associated reference systems, since in a general case two associated reference systems ξ^i and η^i (that is, reference systems whose coordinate lines ξ^4 and η^4 coincide with the world lines of the particles of the medium) are connected by the transformation

$$\eta^\alpha = \eta^\alpha(\xi^\beta), \qquad \eta^1 = \eta^4(\xi^\alpha, \xi^4) \tag{2.5}$$

We shall show that for the system of arguments (2.1) the energy-impulse tensor $P_i^{\ j}$ and 'double' forces $P_i^{\ jk}$, defined by Equations (1.9) and (1.10), are independent of the choice of the associated reference system (let us recall that equations (1.9) and (1.10) contain tensor $\nabla_k x^i_{\ p}$, which is dependent on the choice of the coordinate system).

For this purpose we shall have to determine

$$\frac{\partial g^\wedge_{pq}}{\partial x^i_{\ s}}x^j_{\ s}, \quad \frac{\partial g^\wedge_{pq}}{\partial \nabla_k x^l_{\ s}}, \quad \frac{\partial u^\wedge_p}{\partial x^i_{\ s}}x^j_{\ s}, \quad \frac{\partial u^\wedge_p}{\partial \nabla_k x^l_{\ s}}, \quad \frac{\partial \nabla^\wedge_r u^\wedge_p}{\partial x^i_{\ s}}, \quad \frac{\partial \nabla^\wedge_r u^\wedge_p}{\partial \nabla_k x^l_{\ s}}, \quad \frac{\partial \nabla^\wedge_r u^\wedge_p}{\partial \nabla_k x^i_{\ s}}x^j_{\ s}$$

$$\frac{\partial \nabla^\wedge_r g^\circ_{pq}}{\partial x^i_{\ s}}x^j_{\ s}, \quad \frac{\partial \nabla^\wedge_r g^\circ_{pq}}{\partial \nabla_k x^l_{\ s}}, \quad \frac{\partial \nabla^\wedge_r g^\circ_{pq}}{\partial \nabla_k x^i_{\ s}}x^j_{\ s}$$

$$\frac{\partial \nabla^\wedge_r u^\circ_p}{\partial x^i_{\ s}}x^j_{\ s}, \quad \frac{\partial \nabla^\wedge_r u^\circ_p}{\partial \nabla_k x^l_{\ s}} \quad \frac{\partial \nabla^\wedge_r u^\circ_p}{\partial \nabla_k x^i_{\ s}}x^j_{\ s}$$

Taking into account (2.3) we obtain

$$\frac{\partial g^\wedge_{pq}}{\partial x^i_{\ s}}x^j_{\ s} = x^j_{\ s}\frac{\partial}{\partial x^i_{\ s}}(g_{lm}x^l_{\ p}x^m_{\ q}) = g_{lm}(\delta_i^{\ l}x^j_{\ p}x^m_{\ q} + \delta_i^{\ m}x^j_{\ q}x^l_{\ p})$$

$$\frac{\partial u^\wedge_p}{\partial x^i_{\ s}}x^j_{\ s} = x^j_{\ s}\frac{\partial}{\partial x^i_{\ s}}\left(\frac{g^\wedge_{p4}}{\sqrt{g^\wedge_{44}}}\right) = g_{li}(x^l_{\ p}u^j + x^j_{\ p}u^l) - u^\wedge_p u_i u^j \tag{2.6}$$

$$\frac{\partial g^\wedge_{pq}}{\partial \nabla_k x^l_{\ s}} = \frac{\partial u^\wedge_p}{\partial \nabla_k x^l_{\ s}} = 0$$

To simplify the final formulas, we shall further assume that ξ^4 is the length of the arc along the corresponding world line*, with $u^j = x^j_{\ 4}$, $g^\wedge_{44} = +1$.

(see footnote on the next page)

In determining the derivatives of $\nabla\,\hat{}_r\, u\hat{}_p$, $\nabla\hat{}_r\, g^\circ_{pq}$, and $\nabla\hat{}_r\, u^\circ_p$ with respect to x^i_s and $\nabla_k x^l_s$, use should be made of the formula

$$\nabla\hat{}_r u\hat{}_p = \nabla_{mi}\, u_n\, x^m_r\, x^n_p$$

and it should be taken into account that $\nabla\hat{}_r\, g^\circ_{pq}$, and $\nabla\hat{}_r\, u^\circ_p$ are dependent upon x^i_s and $\nabla_k x^l_s$ only by virtue of $\Gamma\hat{}_{lm}{}^h$, with the latter being defined by the relationship

$$\Gamma\hat{}_{lm}{}^h = \Gamma_{pk}{}^n x^p{}_l x^k{}_m \xi^h{}_n + \frac{\partial^2 x^n}{\partial\xi^l\partial\xi^m}\,\xi^h{}_n = \xi^h{}_n x^k{}_m\left(\frac{\partial}{\partial x^k}\,x^n{}_l + \Gamma_{pk}{}^n x^p{}_l\right) = \tag{2.7}$$
$$= \xi^h{}_n x^k{}_m \nabla_k x^n{}_l = \xi^h{}_n x^k{}_l \nabla_k x^n{}_m$$

where ξ^h_n denotes the partial derivatives of ξ^h with respect to x^n; thus $\xi^h{}_n = \partial\xi^h / \partial x^n$. Using (2.7) and the formula**

$$\frac{\partial\nabla_m u_n}{\partial x^i{}_s}\,x^j{}_s = u^j\nabla_m\gamma_{ni}, \qquad \frac{\partial\nabla_m u_n}{\partial\nabla_k x^l{}_s} = \delta_m{}^k u\hat{}^s\gamma_{nl}$$
$$\frac{\partial\nabla_m u_n}{\partial\nabla_k x^i{}_s}\,x^j{}_s = \delta_m{}^k u^j\gamma_{nl} \quad (\gamma_{nl} = g_{nl} - u_n u_l) \tag{2.8}$$

we find***

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footnote from previous page

* Regardless of the fact that with such restriction upon ξ^4 the relationships $u^j = x^j{}_4$ holds, the following inequalities apply

$$\delta u^j \neq \delta x^j{}_4, \qquad \frac{\partial u^k}{\partial x^i{}_s}\,x^j{}_s \neq \frac{\partial x^k{}_4}{\partial x^i{}_s}\,x^j{}_s$$

This is because after variation of the world lines, ξ^4 will no longer be the length of the arc. A condition that the coordinate ξ^4 should represent the length of the arc even after variation of the world lines would impose undesirable constraints on the variations

$$u^i u^j \nabla_i \delta x_j = 0$$

** Let us note that Equations (2.8) and, consequently, the first three relationships from Equations (2.9) will have the same form in a metric with sign convention (+++−) if we stipulate that

$$\gamma_{nl} = g_{nl} + u_n u_l$$

*** Here, it has also been taken into account that

$$(\partial\xi^h{}_n / \partial x^i{}_s)\,x^j{}_s = -\xi^h{}_i\delta_n{}^j$$

Indeed, if x denotes the determinant of matrix $||x^i{}_j||$, then

$$\frac{\partial\xi^h{}_n}{\partial x^i{}_s}\,x^j{}_s = x^j{}_s\,\frac{\partial}{\partial x^i{}_s}\left(\frac{1}{x}\,\frac{\partial x}{\partial x^n{}_h}\right) = \frac{1}{x}\,\frac{\partial}{\partial x^n{}_h}\left(\frac{\partial x}{\partial x^i{}_s}\,x^j{}_s\right) - \frac{1}{x}\,\frac{\partial x}{\partial x^i{}_s}\,\frac{\partial x^j{}_s}{\partial x^n{}_h} -$$
$$-\frac{1}{x^2}\,\frac{\partial x}{\partial x^n{}_h}\,x^j{}_s\,\frac{\partial x}{\partial x^i{}_s} = \frac{1}{x}\,\delta_i{}^j\,\frac{\partial x}{\partial x^n{}_h} - \xi^h{}_i\delta_n{}^j - \xi^h{}_n x^j{}_s\xi^s{}_i = -\xi^h{}_i\delta_n{}^j$$

$$\frac{\partial \nabla^{\wedge}{}_r u^{\wedge}{}_p}{\partial x^i{}_s} x^j{}_s = u^j \nabla_m \gamma_{ni} x^m{}_r x^n{}_p + \nabla_m u_n (\delta_i{}^m x^j{}_r x^n{}_p + \delta_i{}^n x^j{}_p x^m{}_r)$$

$$\frac{\partial \nabla^{\wedge}{}_r u^{\wedge}{}_p}{\partial \nabla_k x^l{}_s} = u^{\wedge s} \gamma_{nl} x^k{}_r x^n{}_p, \qquad \frac{\partial \nabla^{\wedge}{}_r u^{\wedge}{}_p}{\partial \nabla_k x^i{}_s} x^j{}_s = u^j \gamma_{ni} x^k{}_r x^n{}_p$$

$$\frac{\partial \nabla^{\wedge}{}_r g^{\circ}{}_{pq}}{\partial x^i{}_s} x^j{}_s = \frac{\partial \nabla^{\wedge}{}_r g^{\circ}{}_{pq}}{\partial \Gamma^{\wedge}{}_{lm}{}^h} \frac{\partial \Gamma^{\wedge}{}_{lm}{}^h}{\partial x^i{}_s} x^j{}_s =$$

$$= -(g^{\circ}{}_{mq}\delta_p{}^s + g^{\circ}{}_{mp}\delta_q{}^s)(x^j{}_r \xi^m{}_l \delta_i{}^k - x^k{}_r \xi^m{}_i \delta_l{}^j) \nabla_k x^l{}_s$$

$$\frac{\partial \nabla^{\wedge}{}_r g^{\circ}{}_{pq}}{\partial \nabla_k x^l{}_s} = \frac{\partial \nabla^{\wedge}{}_r g^{\circ}{}_{pq}}{\partial \Gamma^{\wedge}{}_{nm}{}^h} \frac{\partial \Gamma^{\wedge}{}_{nm}{}^h}{\partial \nabla_k x^l{}_s} = -(g^{\circ}{}_{mq}\delta_p{}^s + g^{\circ}{}_{mp}\delta_q{}^s) x^k{}_r \xi^m{}_l \tag{2.9}$$

$$\frac{\partial \nabla^{\wedge}{}_r g^{\circ}{}_{pq}}{\partial \nabla_k x^i{}_s} x^j{}_s = -(g^{\circ}{}_{mq}\delta_p{}^s + g^{\circ}{}_{mp}\delta_q{}^s) x^k{}_r \xi^m{}_l$$

$$\frac{\partial \nabla^{\wedge}{}_r u^{\circ}{}_p}{\partial x^i{}_s} x^j{}_s = \frac{\partial \nabla^{\wedge}{}_r u^{\circ}{}_p}{\partial \Gamma^{\wedge}{}_{lm}{}^h} \frac{\partial \Gamma^{\wedge}{}_{lm}{}^h}{\partial x^i{}_s} x^j{}_s = -u^{\circ}{}_m \delta_p{}^s (x^j{}_r \xi^m{}_l \delta_i{}^k - x^k{}_r \xi^m{}_i \delta_l{}^j) \nabla_k x^l{}_s$$

$$\frac{\partial \nabla^{\wedge}{}_r u^{\circ}{}_p}{\partial \nabla_k x^l{}_s} = \frac{\partial \nabla^{\wedge}{}_r u^{\circ}{}_p}{\partial \Gamma^{\wedge}{}_{nm}{}^h} \frac{\partial \Gamma^{\wedge}{}_{nm}{}^h}{\partial \nabla_k x^l{}_s} = -u^{\circ}{}_m \delta_p{}^s x^k{}_r \xi^m{}_l, \qquad \frac{\partial \nabla^{\wedge}{}_r u^{\circ}{}_p}{\partial \nabla_k x^i{}_s} x^j{}_s = -u^{\circ}{}_m \xi^m{}_i x^k{}_r x^j{}_p$$

Substituting (2.6), (2.8) and (2.9) into (1.9) and (1.10), we obtain

$$P_{(m)i}{}^j = -x^j{}_s \frac{\partial \Lambda_m}{\partial x^l{}_s} - \frac{\partial \Lambda_m}{\partial \nabla_k x^l{}_s} (\delta_i{}^l \nabla_k x^j{}_s - \delta_k{}^j \nabla_i x^l{}_s) - \nabla_k P_i{}^{jk} - \Lambda_m \delta_i{}^j =$$

$$= -\frac{\partial \Lambda_m}{\partial g^{\wedge}{}_{pq}} \frac{\partial g^{\wedge}{}_{pq}}{\partial x^i{}_s} x^j{}_s - \frac{\partial \Lambda_m}{\partial u^{\wedge}{}_p} \frac{\partial u^{\wedge}{}_p}{\partial x^i{}_s} x^j{}_s - \frac{\partial \Lambda_m}{\partial \nabla^{\wedge}{}_r u^{\wedge}{}_p} \frac{\partial \nabla^{\wedge}{}_r u^{\wedge}{}_p}{\partial x^i{}_s} x^j{}_s -$$

$$- \frac{\partial \Lambda_m}{\partial \nabla^{\wedge}{}_r g^{\circ}{}_{pq}} \frac{\partial \nabla^{\wedge}{}_r g^{\circ}{}_{pq}}{\partial x^i{}_s} x^j{}_s - \frac{\partial \Lambda_m}{\partial \nabla^{\wedge}{}_r u^{\circ}{}_p} \frac{\partial \nabla^{\wedge}{}_r u^{\circ}{}_p}{\partial x^i{}_s} x^j{}_s - \left(\frac{\partial \Lambda_m}{\partial \nabla^{\wedge}{}_r u^{\wedge}{}_p} \frac{\partial \nabla^{\wedge}{}_r u^{\wedge}{}_p}{\partial \nabla_k x^l{}_s} + \right.$$

$$\left. + \frac{\partial \Lambda_m}{\partial \nabla^{\wedge}{}_r g^{\circ}{}_{pq}} \frac{\partial \nabla^{\wedge}{}_r g^{\circ}{}_{pq}}{\partial \nabla_k x^l{}_s} + \frac{\partial \Lambda_m}{\partial \nabla^{\wedge}{}_r u^{\circ}{}_p} \frac{\partial \nabla^{\wedge}{}_r u^{\circ}{}_p}{\partial \nabla_k x^l{}_s} \right) (\delta_i{}^l \nabla_k x^j{}_s - \delta_k{}^j \nabla_i x^l{}_s) -$$

$$- \nabla_k P_i{}^{jk} - \Lambda_m \delta_i{}^j = -\frac{\partial \Lambda_m}{\partial g^{\wedge}{}_{pq}} g_{lm} (\delta_i{}^l x^j{}_p x^m{}_q + \delta_i{}^m x^j{}_q x^l{}_p) -$$

$$- \frac{\partial \Lambda_m}{\partial u^{\wedge}{}_p} [g_{li}(x^l{}_p u^j + x^j{}_p u^l) - u^{\wedge}{}_p u_i u^j] - \frac{\partial \Lambda_m}{\partial \nabla^{\wedge}{}_r u^{\wedge}{}_p} [u^j \nabla_m \gamma_{ni} x^m{}_r x^n{}_p + \tag{2.10}$$

$$+ \nabla_m u_n (\delta_i{}^m x^j{}_r x^n{}_p + \delta_i{}^n x^j{}_p x^m{}_r)] + \frac{\partial \Lambda_m}{\partial \nabla^{\wedge}{}_r g^{\circ}{}_{pq}} (g^{\circ}{}_{mq}\delta_p{}^s + g^{\circ}{}_{mp}\delta_q{}^s)(x^j{}_r \xi^m{}_l \delta_i{}^k -$$

$$- x^k{}_r \xi^m{}_i \delta_l{}^j) \nabla_k x^l{}_s + \frac{\partial \Lambda_m}{\partial \nabla^{\wedge}{}_r u^{\circ}{}_p} u^{\circ}{}_m \delta_p{}^s (x^j{}_r \xi^m{}_l \delta_i{}^k - x^k{}_r \xi^m{}_i \delta_l{}^j) \nabla_k x^l{}_s -$$

$$- \left[\frac{\partial \Lambda_m}{\partial \nabla^{\wedge}{}_r u^{\wedge}{}_p} u^{\wedge s} \gamma_{nl} x^k{}_r x^n{}_p - \frac{\partial \Lambda_m}{\partial \nabla^{\wedge}{}_r g^{\circ}{}_{pq}} (g^{\circ}{}_{mq}\delta_p{}^s + g^{\circ}{}_{mp}\delta_q{}^s) x^k{}_r \xi^m{}_l - \right.$$

$$\left. - \frac{\partial \Lambda_m}{\partial \nabla^{\wedge}{}_r u^{\circ}{}_p} u^{\circ}{}_m \xi^m{}_l x^k{}_r \delta_p{}^s \right] (\delta_i{}^l \nabla_k x^j{}_s - \delta_k{}^j \nabla_i x^l{}_s) - \nabla_k P_i{}^{jk} - \Lambda_m \delta_i{}^j$$

$$P_i^{\ jk} = -\frac{1}{2}\left[\frac{\partial\Lambda_m}{\partial\nabla_k x^i_{\ s}} x^j_{\ s} + (j\leftrightarrow k)\right] = -\frac{1}{2}\left[\left(\frac{\partial\Lambda_m}{\partial\nabla^\wedge_{\ r}u^\wedge_{\ p}}\frac{\partial\nabla^\wedge_{\ r}u^\wedge_{\ p}}{\partial\nabla_k x^i_{\ s}} x^j_{\ s} + \right.\right.$$
$$\left.\left. + \frac{\partial\Lambda_m}{\partial\nabla^\wedge_{\ r}g^\circ_{\ pq}}\frac{\partial\nabla^\wedge_{\ r}g^\circ_{\ pq}}{\partial\nabla_k x^i_{\ s}} x^j_{\ s} + \frac{\partial\Lambda_m}{\partial\nabla^\wedge_{\ r}u^\circ_{\ p}}\frac{\partial\nabla^\wedge_{\ r}u^\circ_{\ n}}{\partial\nabla_k x^i_{\ s}} x^j_{\ s}\right) + (j\leftrightarrow k)\right] =$$
$$= -\frac{1}{2}\left[\left(\frac{\partial\Lambda_m}{\partial\nabla^\wedge_{\ r}u^\wedge_{\ p}} u^q\gamma_{ln}\xi^s_{\ q}x^l_{\ m}x^n_{\ p} - \frac{\partial\Lambda_m}{\partial\nabla^\wedge_{\ r}g^\circ_{\ pq}}(g^\circ_{\ mq}\delta_p^{\ s} + \right.\right. \tag{2.11}$$
$$\left.\left. + g^\circ_{\ mp}\delta_q^{\ s}) - \frac{\partial\Lambda_m}{\partial\nabla^\wedge_{\ r}u^\circ_{\ s}} u^\circ_{\ m}\right)\xi^m_{\ i}x^j_{\ s}x^k_{\ r} + (j\leftrightarrow k)\right]$$

All the terms containing $\nabla_k x_a^{\ i}$, cancel as expected. This confirms the fact that for the system of arguments (2.1), the tensors $P_{(m)i}^{\ j}$ and P_i^{jk} are independent of the choice of the associated system of reference.

Equations (2.10) and (2.11) give in the associated frame of reference, the following expressions for the contravariant components of the canonical tensor of energy impulse $P_{(\hat{m})}^{\ ij}$ and for the tensor $P^{\wedge ijk}$ associated with the internal angular momentum and the internal surface momentums*

$$P_{(\hat{m})}^{\ ij} = -2\frac{\partial\Lambda_m}{\partial g^\wedge_{\ ij}} - \left(u^{\wedge i}\frac{\partial\Lambda_m}{\partial u^\wedge_{\ j}} + u^{\wedge j}\frac{\partial\Lambda_m}{\partial u^\wedge_{\ i}}\right) + \left(u^\wedge_{\ n}\frac{\partial\Lambda_m}{\partial u^\wedge_{\ p}}\right)u^{\wedge i}u^{\wedge j} -$$
$$- \frac{\partial\Lambda_m}{\partial\nabla^\wedge_{\ r}u^\wedge_{\ p}}\nabla^\wedge_{\ r}(u^{\wedge j}\gamma^\wedge_{\ p}{}^i + u^{\wedge i}\delta_p^{\ j}) - \nabla^\wedge_{\ k}P^{\wedge ijk} - \Lambda_m g^{\wedge ij}$$
$$P^{\wedge ijk} = -\frac{1}{2}\left[\left(\frac{\partial\Lambda_m}{\partial\nabla^\wedge_{\ k}u^\wedge_{\ p}}u^{\wedge j}\gamma^\wedge_{\ p}{}^i - 2g^{\wedge ip}g^\circ_{\ pq}\frac{\partial\Lambda_m}{\partial\nabla^\wedge_{\ k}g^\circ_{\ qj}} - \right.\right. \tag{2.12}$$
$$\left.\left. - g^{\wedge ip}u^\circ_{\ p}\frac{\partial\Lambda_m}{\partial\nabla^\wedge_{\ k}u^\circ_{\ j}}\right) + (j\leftrightarrow k)\right]$$

Equations of motion (1.17) appear in this case in the form

$$\nabla^\wedge_{\ j}P_{(\hat{m})i}^{\ j} + \Lambda^{\wedge jk}_{\ \ l}R^\wedge_{\ ijk}{}^l = 0$$

$$\Lambda^{\wedge jk}_{\ \ l} = \frac{\partial\Lambda_m}{\partial\nabla_j x^l_{\ s}}x^k_{\ s} + \frac{1}{2}g^\wedge_{\ lp}g^{\wedge qj}\frac{\partial\Lambda_m}{\partial\nabla_k x^q_{\ s}}x^p_{\ s} = u^{\wedge k}\gamma^\wedge_{\ lp}\frac{\partial\Lambda_m}{\partial\nabla^\wedge_{\ j}u^\wedge_{\ p}} -$$
$$- 2g^\circ_{\ lq}\frac{\partial\Lambda_m}{\partial\nabla^\wedge_{\ j}g^\circ_{\ qk}} - u^\circ_{\ l}\frac{\partial\Lambda_m}{\partial\nabla^\wedge_{\ j}u^\circ_{\ k}} + \frac{1}{2}u^\wedge_{\ l}\gamma^\wedge_{\ p}{}^j\frac{\partial\Lambda_m}{\partial\nabla^\wedge_{\ k}u^\wedge_{\ p}} - \tag{2.13}$$
$$- g^{\wedge js}g^\wedge_{\ lp}g^\circ_{\ sq}\frac{\partial\Lambda_m}{\partial\nabla^\wedge_{\ k}g^\circ_{\ pq}} - \frac{1}{2}g^{\wedge js}g^\wedge_{\ lp}u^\circ_{\ s}\frac{\partial\Lambda_m}{\partial\nabla^\wedge_{\ r}u^\circ_{\ p}}$$

3. A generalisation of the model of an elastic body. As an example we shall consider, within the framework of the Special Theory of Relativity ($\Lambda = \Lambda_m$), a system of equations describing the model of an elastic body whose characteristic parameters include derivatives with respect to time and coordinates of the tensor of finite deformations. As we know [5], in this case it is nor possible to derive all the equations of state from the first law of thermodynamics and it is necessary to use the variational principle.

* In the derivation of Equations (2.12) use was made of the identity

$$\nabla^\wedge_{\ i}u^\circ_{\ p} = \gamma^\wedge_{\ pl}\nabla^\wedge_{\ i}u^{\wedge l}$$

Let us first determine the tensor of finite deformations in the Special Theory (and analogously, in the General Theory of Relativity). In defining this tensor, it is necessary to remember that the distance in a three-dimensional space (that is, the distance between two simultaneous events) which can be defined by means of the quantity γ_{ij} ([3] p. 286)

$$-dl^2 = \gamma_{\alpha\beta}\, dx^\alpha dx^\beta, \qquad \gamma_{ij} = g_{ij} - g_{i4}g_{j4} / g_{44}, \quad \gamma_{i4} = \gamma_{4i} = 0$$

For the purpose of measurement of three-dimensional distances in the associated reference system tensor [4]

$$\gamma\hat{}_{pq} = g\hat{}_{pq} - u\hat{}_p u\hat{}_q$$

can be constructed, where $\gamma\hat{}_{pq}$, by virtue of Equations (2.3), coincides with $g\hat{}_{pq} - g\hat{}_{p4}g\hat{}_{q4} / g\hat{}_{44}$ in the system ξ^{i*}. In a system of coordinates which is not an associated system, components of the tensor $g_{pq} - u_p u_q$, naturally no longer define a three-dimensional distance. For the purpose of description of a deformation it is necessary to equate the element of length dl^2 (ξ^α, ξ^4, $d\xi^\alpha$) with the element of length in the initial state, that is with dl^2 (ξ^α, $\xi_0{}^4$, $d\xi^\alpha$) $= dl_0{}^2$ (indeed, an element of length between two simultaneous events in the associated reference system is a four-dimensional invariant $-dl^2 = \gamma\hat{}_{\alpha\beta}d\xi^\alpha d\xi^\beta = \gamma\hat{}_{pq}d\xi^p d\xi^q$). The element of length in the initial state is described by the tensor field $\gamma_{pq}{}^\circ$

$$dl_0{}^2 = \gamma_{pq}{}^\circ\, d\xi^p d\xi^q, \ \ \gamma_{pq}{}^\circ\,(\xi^\alpha, \xi^4) = \gamma\hat{}_{pq}\,(\xi^\alpha, \xi_0{}^4) = g_{pq}{}^\circ - u_p{}^\circ u_q{}^\circ$$
$$\xi_0{}^4 = \text{const}, \quad \nabla\hat{}_4\gamma_{pq}{}^\circ \neq 0, \qquad \gamma_{4p}{}^\circ = \gamma_{p4}{}^\circ = 0 \tag{3.1}$$

The tensor of finite deformations $E\hat{}_{pq}$ is constructed as follows**:-

$$\frac{1}{2}\,(dl^2 - dl_0{}^2) = E\hat{}_{pq}d\xi^p d\xi^q, \quad E\hat{}_{pq} = -\frac{1}{2}\,(\gamma\hat{}_{pq} - \gamma^\circ{}_{pq}), \quad E\hat{}_{p4} = E\hat{}_{4p} = 0$$

Let us note that the tensor field $\gamma_{pq}{}^\circ$, formed according to (3.1) does not depend on the particular associated system in which it was formulated (by virtue of (2.5) and of relationship $\gamma\hat{}_{p4} = \gamma\hat{}_{4p} = 0$). Therefore the present determination of the finite deformations is the expected one.

Let us now assume that the defining parameters include the quantities

$$g^{\circ pq}, \quad u^{\circ p}, \quad u\hat{}_p, \quad E\hat{}_{pq}, \quad \nabla\hat{}_r E\hat{}_{pq}, \quad K\hat{}^C$$

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* In the metric with a sign convention (+++−) $\gamma\hat{}_{pq} = g\hat{}_{pq} + u\hat{}_p u\hat{}_q$, since in that case equations (2.3) can alternatively be written

$$u\hat{}^p = \frac{\delta_4{}^p}{\sqrt{-g\hat{}_{44}}}\,, \qquad u\hat{}_p = \frac{g\hat{}_{p4}}{\sqrt{-g\hat{}_{44}}}$$

** In the present analysis the following sign convention is used for metric (− − − +). For the metric (+++−) the tensor of finite deformation is expressed in the form

$$E\hat{}_{pq} = 1/2\left(\gamma\hat{}_{pq} - \gamma^\circ{}_{pq}\right)$$

The case when the arguments of Λ include $g^{\circ pq}$, $E^{\wedge}{}_{pq}$, and $K^{\wedge C}$, was considered in the Special Theory of Relativity by Sedov [1] while the analogous problem in the Newtonian mechanics is a classical one. The model, considered in [8] within the framework of Newtonian mechanics, had amongst its parameters the space derivatives of the tensor of finite deformations. The present model is a model of an elastic body whose Lagrangian depends upon the first time derivatives of the defining parameters.

Equations (1.9) and (1.10), (or, for the present model (2.12)), are the required equations of state. By

$$\frac{\partial E^{\wedge}{}_{ls}}{\partial g^{\wedge}{}_{pq}} = -\frac{1}{4}(\delta_p{}^l\delta_q{}^s + \delta_p{}^s\delta_q{}^l), \qquad \frac{\partial E^{\wedge}{}_{ls}}{\partial u^{\wedge}{}_p} = \frac{1}{2}(\delta_l{}^p u^{\wedge}{}_s + \delta_s{}^p u^{\wedge}{}_l)$$

$$\frac{\partial \nabla^{\wedge}{}_m E^{\wedge}{}_{ls}}{\partial g^{\wedge}{}_{pq}} = 0, \qquad \frac{\partial \nabla^{\wedge}{}_m E^{\wedge}{}_{ls}}{\partial u^{\wedge}{}_p} = \frac{1}{2}(\delta_s{}^p \nabla^{\wedge}{}_m u^{\wedge}{}_l + \delta_l{}^p \nabla^{\wedge}{}_m u^{\wedge}{}_s).$$

$$\frac{\partial \nabla^{\wedge}{}_m E^{\wedge}{}_{ls}}{\partial \nabla^{\wedge}{}_r u^{\wedge}{}_p} = \frac{1}{2}\delta_m{}^r(\delta_l{}^p u^{\wedge}{}_s + \delta_s{}^p u^{\wedge}{}_l)$$

$$\frac{\partial \nabla^{\wedge}{}_m E^{\wedge}{}_{ls}}{\partial \nabla^{\wedge}{}_r g^{\circ}{}_{pq}} = \frac{1}{4}\delta_m{}^r(\delta_l{}^p\delta_s{}^q + \delta_s{}^p\delta_l{}^q), \qquad \frac{\partial \nabla^{\wedge}{}_m E^{\wedge}{}_{ls}}{\partial \nabla^{\wedge}{}_r u^{\circ}{}_p} = -\frac{1}{2}\delta_m{}^r(\delta_l{}^p u^{\circ}{}_s + \delta_s{}^p u^{\circ}{}_l)$$

we obtains from Formulas (2.12), the following expressions for the contravariant components of the tensor of the energy impulse $P^{\wedge}_{(m)}{}^{ij}$ and of the tensor $P^{\wedge ijk}$

$$P^{\wedge}_{(m)}{}^{ij} = P^{\wedge ij} = \frac{\partial \Lambda}{\partial E^{\wedge}{}_{ij}} - u^{\wedge i}\left(\frac{\partial \Lambda}{\partial u^{\wedge}{}_j} + u^{\wedge}{}_q \frac{\partial \Lambda}{\partial E^{\wedge}{}_{qj}}\right) - u^{\wedge j}\left(\frac{\partial \Lambda}{\partial u^{\wedge}{}_i} + u^{\wedge}{}_q \frac{\partial \Lambda}{\partial E^{\wedge}{}_{qi}}\right) +$$

$$+ u^{\wedge i}u^{\wedge j}\left(u^{\wedge}{}_p \frac{\partial \Lambda}{\partial u^{\wedge}{}_p} + u^{\wedge}{}_p u^{\wedge}{}_q \frac{\partial \Lambda}{\partial E^{\wedge}{}_{pq}}\right) + \frac{\partial \Lambda}{\partial \nabla^{\wedge}{}_r E^{\wedge}{}_{pq}} \nabla^{\wedge}{}_r(\gamma^{\wedge}{}_p{}^i \gamma^{\wedge}{}_q{}^j) -$$

$$- \nabla^{\wedge}{}_k P^{\wedge ijk} - \Lambda g^{\wedge ij}$$

$$P^{\wedge ijk} = -\frac{1}{2}\left[(u^{\wedge}{}_p u^{\wedge j}\gamma^{\wedge}{}_p{}^i - \delta_p{}^j g^{\wedge is}\gamma^{\circ}{}_{sq}) \frac{\partial \Lambda}{\partial \nabla^{\wedge}{}_k E^{\wedge}{}_{pq}} + (j \leftrightarrow k)\right]$$

After simple transformations, we obtain for the components of the tensor $P_i^{\wedge j}$ with mixed indices,

$$P^{\wedge}{}_i{}^j = \gamma^{\wedge}{}_{ip}\gamma^{\wedge}{}_q{}^j\left(\frac{\partial \Lambda}{\partial E^{\wedge}{}_{pq}} - \nabla^{\wedge}{}_k \frac{\partial \Lambda}{\partial \nabla^{\wedge}{}_k E^{\wedge}{}_{pq}}\right) - (\gamma^{\wedge}{}_{ip}u^{\wedge j} + \delta_p{}^j u^{\wedge}{}_i)\frac{\partial \Lambda}{\partial u^{\wedge}{}_p} +$$

$$+ \nabla^{\wedge}{}_k\left(\gamma^{\wedge}{}_{ip}\gamma^{\wedge}{}_q{}^j \frac{\partial \Lambda}{\partial \nabla^{\wedge}{}_k E^{\wedge}{}_{pq}} - P^{\wedge}{}_i{}^{jk}\right) - \Lambda\delta_i{}^j$$

$$P^{\wedge}{}_i{}^{jk} = \frac{1}{2}\gamma^{\wedge}{}_{iq}\left(\gamma^{\wedge}{}_p{}^j \frac{\partial \Lambda}{\partial \nabla^{\wedge}{}_k E^{\wedge}{}_{pq}} + \gamma^{\wedge}{}_p{}^k \frac{\partial \Lambda}{\partial \nabla^{\wedge}{}_j E^{\wedge}{}_{pq}}\right) + \tag{3.2}$$

$$+ E^{\wedge}{}_{iq}\left(\frac{\partial \Lambda}{\partial \nabla^{\wedge}{}_k E^{\wedge}{}_{qj}} + \frac{\partial \Lambda}{\partial \nabla^{\wedge}{}_j E^{\wedge}{}_{qk}}\right)$$

The tensor of the energy impulse is in this case anti symmetric. Introducing the density ρ according to

$$\rho = \rho_0 \sqrt{\gamma^{\circ}} / \sqrt{\gamma^{\wedge}}, \qquad \gamma^{\circ} = \det\|\gamma^{\circ}{}_{\alpha\beta}\|, \quad \gamma^{\wedge} = \det\|\gamma^{\wedge}{}_{\alpha\beta}\| \tag{3.3}$$

and bearing in mind the relationships

$$g^{\wedge\alpha\beta}\gamma^{\wedge}{}_{\beta\sigma} = \delta_\sigma{}^\alpha, \qquad \partial\rho \,/\, \partial E^{\wedge}{}_{\alpha\beta} = -\,2\partial\rho \,/\, \partial\gamma^{\wedge}{}_{\alpha\beta} = \rho g^{\alpha\beta}$$

we can write the equations for the spatial part of the energy impulse tensor in the form

$$P^{\wedge\alpha\beta} = \rho\,\frac{\partial\,(\Lambda/\rho)}{\partial E^{\wedge}{}_{\alpha\beta}} - \nabla^{\wedge}{}_k P^{\wedge\alpha\beta k} - \frac{\partial\Lambda}{\partial\nabla^{\wedge}{}_k E^{\wedge}{}_{pq}}\,\nabla^{\wedge}{}_k\,(\gamma^{\wedge}{}_p{}^{\alpha}\gamma^{\wedge}{}_q{}^{\beta}) \tag{3.4}$$

Apart from the associated reference system we shall introduce at every point of the space-time continuum a so called proper system of coordinates, which is an ortho-normalized rectilinear reference system in which the element of a medium present at a given point of the space-time continuum, is at rest. The quantities computed with respect to this coordinate system will be identified by an asterisk. It is known that in the proper coordinate system the spatial part of the energy impulse tensor $P^{*\alpha\beta}$ has the meaning of a stress tensor taken with an opposite sign

$$P^{*\alpha\beta} = -\,p^{\alpha\beta} \tag{3.5}$$

Here $p^{\alpha\beta}$ is the stress tensor. In this way, equations (3.4) written for the proper system of coordinates give the expression for the stress tensor. In particular, if $\Lambda = \Lambda\,(g^{\circ pq},\ u^{\circ p},\ E^{\wedge}{}_{pq})$ (a classical elastic body), then

$$p^{\alpha\beta} = -\,\rho\,\frac{\partial\,(\Lambda/\rho)}{\partial E^{*}{}_{\alpha\beta}} \tag{3.6}$$

Let us now postulate that in the proper system of coordinates the laws of thermodynamics (and consequently the thermodynamic relationships) should have the same from as in the nonrelativistic mechanics. Then

$$p^{\alpha\beta} = \rho\,\frac{\partial U}{\partial E^{*}{}_{\alpha\beta}} \tag{3.7}$$

Where U is the internal energy per unit mass. Comparison of Equations (3.6) and (3.7) gives

$$\Lambda = -\,\rho U$$

It also follows from (3.2) that $P^{\wedge}{}_4{}^4 = \rho U$.

In this particular case the stress tensor is symmetric. However, if the arguments of Λ include the gradients of the tensor of finite deformations, then $p^{\alpha\beta} \neq p^{\beta\alpha}$ since $P^{\alpha\beta k} \neq P^{\beta\alpha k}$. It should also be noted that the stress tensor is a linear function of acceleration of the particles of the medium. This phenomenon has no analogy in nonrelativistic mechanics.

For the purpose of defining the components of the tensor of stresses within the framework of the Newtonian mechanics, it is sufficient to perform a formal operation in Equation (3.4), namely*

$$\gamma^{\wedge}{}_{\alpha\beta} = g^{\wedge}{}_{\alpha\beta}, \qquad \gamma^{\wedge\alpha}{}_{\beta} = \delta_\beta{}^\alpha, \qquad \gamma^{\wedge}{}_\alpha{}^4 = 0$$

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* This follows from the equations

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$$\delta E^{\wedge}{}_{pq} = {}^1\!/_2 \left(\gamma^{\wedge}{}_p{}^i \gamma_q{}^j + \gamma^{\wedge}{}_q{}^i \gamma^{\wedge}{}_p{}^j\right) \nabla^{\wedge}{}_j (\delta x^i)^{\wedge}$$

$$\delta \nabla^{\wedge}{}_r E^{\wedge}{}_{pq} = {}^1\!/_2 \left[\nabla^{\wedge}{}_r \left(\gamma^{\wedge}{}_p{}^i \gamma^{\wedge}{}_q{}^j + \gamma^{\wedge}{}_q{}^i \gamma^{\wedge}{}_p{}^j\right)\right] \nabla^{\wedge}{}_j (\delta x_i)^{\wedge} + \tag{3.8}$$
$$+ \left[{}^1\!/_2 \left(\gamma^{\wedge}{}_p{}^i \gamma_q{}^j + \gamma^{\wedge}{}_q{}^i \gamma^{\wedge}{}_p{}^j\right) \rightarrow \left(E^{\wedge}{}_p{}^i \delta_q{}^j + E^{\wedge}{}_q{}^i \delta_p{}^j\right)\right] \nabla^{\wedge}{}_r \nabla^{\wedge}{}_j (\delta x_i)^{\wedge}$$

Indeed, if (3.4) is to give an expression for the stress tensor in nonrelativistic mechanics, it is necessary that $\delta E^{\wedge}{}_{pq}$ and $\delta \nabla^{\wedge}{}_r E^{\wedge}{}_{pq}$ coincide at p, $q = 1, 2, 3$ and $r = 1, 2, 3, 4$ with the variations of the Newtonian tensor of deformations $\varepsilon^{\wedge}{}_{\alpha\beta} = (g^{\wedge}{}_{\alpha\beta} - g^{\circ}{}_{\alpha\beta}) / 2$ and of its gradients; $g^{\wedge}{}_{\alpha\beta}$ is here a metric tensor in the associated reference system in a normal space, and obviously $E^{\wedge}{}_{\alpha\beta} = \varepsilon^{\wedge}{}_{\alpha\beta}$. If we note that the following relationships are valid:

$$\delta \varepsilon^{\wedge}{}_{\alpha\beta} = {}^1\!/_2 \left(\delta_\alpha{}^\sigma \delta_\beta{}^\omega + \delta_\beta{}^\sigma \delta_\alpha{}^\omega\right) \nabla^{\wedge}{}_\sigma (\delta x_\omega)^{\wedge}$$

$$\delta \nabla^{\wedge}{}_\gamma \varepsilon^{\wedge}{}_{\alpha\beta} = \left[{}^1\!/_2 \left(\delta_\alpha{}^\sigma \delta_\beta{}^\omega + \delta_\beta{}^\sigma \delta_\alpha{}^\omega\right) - \left(\varepsilon^{\wedge}{}_\alpha{}^\omega \delta_\beta{}^\sigma + \varepsilon^{\wedge}{}_\beta{}^\omega \delta_\alpha{}^\sigma\right)\right] \nabla^{\wedge}{}_\gamma \nabla^{\wedge}{}_\sigma (\delta x_\omega)^{\wedge},$$

$$\delta \nabla^{\wedge}{}_4 \varepsilon^{\wedge}{}_{\alpha\beta} = \delta \frac{1}{c} \frac{\partial \varepsilon^{\wedge}{}_{\alpha\beta}}{\partial t} = \frac{1}{2} \left(\delta_\alpha{}^\sigma \delta_\beta{}^\omega + \delta_\beta{}^\sigma \delta_\alpha{}^\omega\right) \nabla^{\wedge}{}_4 \nabla^{\wedge}{}_\sigma (\delta x_\omega)^{\wedge}$$

then it becomes obvious that the formal operation indicated in the text does indeed lead to an expression for the Newtonian stress tensor. Let us note that Formulas (3.8) remain valid only in the case when the metric of the space-time manifold is not subjected to variations $(\partial g_{ij} = 0)$, and the last of them is valid only in space of zero curvature, since it is based upon

$$\delta \frac{\partial}{\partial \xi^q} x^i{}_p = \frac{\partial}{\partial \xi^q} \delta x^i{}_p$$

which ceases to hold when the curvature tensor $R_{ijk}{}^l$ is different from zero.

In that case, the last term of (3.4) becomes zero, and the stress tensor assumes the form

$$p^{\wedge\alpha\beta} = -\rho \frac{\partial (\Lambda/\rho)}{\partial E^{\wedge}{}_{\alpha\beta}} + \nabla^{\wedge}{}_k p^{\wedge\alpha\beta k} \tag{3.9}$$

Comparison of (3.9) for the model of a classical elastic body, with the formula (3.7) written in the associated reference system, shows that in the Newtonian mechanics it is possible to stipulate that $\Lambda = \rho\,(U + f)$ where f is independent of the tensor of finite deformations. If we assume that $\Lambda = \rho\,(U + f)$ where the internal energy U contains the tensor of finite defomations and its spacial derivatives while f remains independent of the latter, then (3.9) becomes identical with the equations of state which follow from the first law of thermodynamics for reversible processes*. If Λ contains also a derivative of

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* In the nonrelativistic mechanics of continuous media, the equations of state follow from the first law of thermodynamics [1 to 5]

$$dU = \frac{1}{\rho} p^{\alpha\beta} \nabla_\beta v_\alpha \, dt + \frac{1}{\rho} dq^{(e)} + \frac{1}{\rho} dq^{**}$$

where $dq^{(e)}$ is the inflow of heat and dq^{**} is an adiabatic flow of energy.

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the tensor of finite deformations with respect to time, then (3.9) contains an additional term $\nabla^{\wedge}{}_4 P^{\wedge\alpha\beta 4}$, which cannot be obtained from the first law of thermodynamics.

Let us now write the expressions for the components of tensor $P^j{}_i$ with a time-component in the proper reference system of one of the particles of the medium. Bearing in mind that in the proper reference system,

$$\gamma^*{}_{\alpha\beta} = g^*{}_{\alpha\beta},\quad \gamma^{*\alpha\beta} = g^{*\alpha\beta},\quad \gamma^{*k4} = \gamma^{*4k} = \gamma^*{}_{k4} = \gamma^*{}_{4k} = \gamma^*{}_k{}^4 = \gamma^*{}_4{}^k = 0$$

$$g^*{}_{11} = g^*{}_{22} = g^*{}_{33} = -g^*{}_{44} = -1,\quad u^{*4} = u^*{}_4 = 1,\quad u^{*\alpha} = u^*{}_\alpha = 0$$

$$\partial u^*{}_4 / \partial x^{*k} = \partial u^{*4} / \partial x^{*k} = 0$$

we have

$$P^*{}_4{}^4 = P^{*44} = P^*{}_{44} = -\Lambda - \frac{\partial\Lambda}{\partial u^*{}_4} + \frac{1}{2}\frac{\partial\Lambda}{\partial(E^*{}_{\alpha\beta}/\partial x^{*4})}\frac{\partial u^*{}_\beta}{\partial x^{*\alpha}} - \tag{310}$$
$$-\frac{\partial}{\partial x^{*k}}\left[E^*{}_q{}^4\left(\frac{\partial\Lambda}{\partial(\partial E^*{}_{q4}/\partial x^{*k})} + \frac{\partial\Lambda}{\partial(\partial E^*{}_{qk}/\partial x^{*4})}\right)\right]$$

$$P^*{}_4{}^\alpha = -\frac{\partial\Lambda}{\partial u^*{}_\alpha} - \frac{1}{2}\frac{\partial\Lambda}{\partial(\partial E^*{}_{\alpha\beta}/\partial x^{*k})}\frac{\partial u^*{}_\beta}{\partial x^{*k}} + \frac{1}{2}\frac{\partial\Lambda}{\partial(\partial E^*{}_{\beta\gamma}/\partial x^{*\alpha})}\frac{\partial u^*{}_\beta}{\partial x^{*\gamma}} -$$
$$-\frac{\partial}{\partial x^{*k}}\left[E^*{}_{q4}\left(\frac{\partial\Lambda}{\partial(\partial E^*{}_{q\alpha}/\partial x^{*k})} + \frac{\partial\Lambda}{\partial(\partial E^*{}_{qk}/\partial x^{*\alpha})}\right)\right]$$

$$P^*{}_\alpha{}^4 = -g_{\alpha\beta}{}^*\left(\frac{\partial\Lambda}{\partial u^*{}_\beta} + \frac{1}{2}\frac{\partial\Lambda}{\partial(\partial E^*{}_{\beta\gamma}/\partial x^{*k})}\frac{\partial u^*{}_\gamma}{\partial x^{*k}}\right) -$$

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Introducing displacements $W_\alpha = v_\alpha dt$, one may write

$$dU = \frac{1}{\rho}p^{\alpha\beta}\nabla_\beta w_\alpha + \frac{1}{\rho}dq^{(e)} + \frac{1}{\rho}dq^{**}$$

The construction of models in the Newtonian mechanics involves various assumptions with refer to the arguments of U and the form of $dq^{(e)}$ and dq^{**}.

In particular, if it is assumed that $U = U(g^{\circ\alpha\beta}E^{\wedge}{}_{\alpha\beta},\ \nabla^{\wedge}{}_\gamma E^{\wedge}{}_{\alpha\beta},\ S)$, that the processes in each small particle are reversible $(\rho^{-1}dq^{(e)} = TdS)$ and that dq^{**} is the flow of energy across the boundary of each small particle $dq^{**} = \nabla_\gamma Q^\gamma$, where $Q^\gamma = Q^{\alpha\beta\gamma}\nabla_\beta w_\alpha$, with $Q^{\alpha\beta\gamma} = Q^{\alpha\gamma\beta}$, then one finds from (3.9) that the corresponding equations of state obtained by the variational principle and from the first law of thermodynamics are identical. It should be noted that the adiabatic flow of energy dq^{**} depends in this case on the angular velocities, whilst the latter do not affect the internal energy. Models for which the internal energy is a function of the tensor of finite deformations and its spatial derivatives, and the quantity dq^{**} is independent of the angular velocities were considered by Idin [9]. In the derivation of the corresponding models by means of the variational principle, it is necessary to stipulate that the function δW^* differs from zero.

$$- \frac{\partial}{\partial x^{*k}} \left[\frac{1}{2} \Upsilon^*{}_{\alpha p} \Upsilon^*{}_q{}^k \frac{\partial \Lambda}{\partial (\partial E^*{}_{pq} / \partial x^{*4})} + E^*{}_{\alpha q} \left(\frac{\partial \Lambda}{\partial (\partial E^*{}_{q4} / \partial x^{*k})} + \frac{\partial \Lambda}{\partial (\partial E^*{}_{qk} / \partial x^{*4})} \right) \right]$$

For the models of a classical elastic body in the Special Theory of Relativity $\Lambda = \Lambda(g^{\circ pq}, u^{\circ p}, E^{\wedge}{}_{pq})$ it was shown that Λ is the density of internal energy (with a reverse sign) which, together with Equation (3.10) shows that $P^*{}_4{}^4 = \rho U$. A study of more complex models, for instance $\Lambda = \Lambda(g^{\circ pq}, u^{\circ p}, u^{\wedge p}, E^{\wedge}{}_{pq})$ or $\Lambda = \Lambda(g^{\circ pq}, u^{\circ p}, E^{\wedge}{}_{pq}, \nabla^{\wedge}{}_r E^{\wedge}{}_{pq})$ will immediately show that $P^*{}_4{}^4 \neq -\Lambda$. For such models, a question obviously arises as to what should be defined as the internal energy density, $P^*{}_4{}^4$ or $(-\Lambda)$.

The quantities $P^*{}_4{}^\alpha$ represent, in spatial transformations, the components of a three-dimensional vector which determines the adiabatic inflow of energy to a particle of the medium. In the case of classical elasticity $\Lambda = \Lambda(g^{\circ pq}, u^{\circ p}, E^{\wedge}{}_{pq})$ there is no adiabatic inflow of energy. If the defining parameters include the gradients of deformations, then this inflow of energy is quite substantial and cannot be ignored. The components of the energy impulse tensor $P^*{}_\alpha{}^4$ and $P^{\wedge}{}_\alpha{}^4$ are, for such models, different from zero. This shows that in the proper reference system in which the particle of the medium is stationary, as well as in the associated reference system in which the whole continuum is at rest, there is, nevertheless, a macroscopic transfer of momentum. This arises because of the adiabatic flow of energy which, as it is known, produces a flow of momentum. Also, the macroscopic transfer of momentum is connected with the presence of an internal moment of momentum (since $P^{*ij} \neq P^{*ji}$). The terms in the expression for $P^*{}_\alpha{}^4$, have no analogues in the nonrelativistic mechanics.

Equations of state (3.2), together with the equations of motion (1.18) form a self-contained system which fully describes the given model of a continuous medium in the Special Theory. In the General Theory, the divergence of the canonical tensor of the energy impulse does not as a rule become equal to zero. $P_{(m)i}{}^j$ will in that case be defined by Equation (1.17) which assumes the following form for the medium under consideration:

$$\nabla^{\wedge}{}_j P^{\wedge}{}_{(m)i}{}^j + \Lambda^{\wedge}{}_l{}^{jk} R^{\wedge}{}_{ijk}{}^l = 0$$

$$\Lambda^{\wedge jk}{}_l = u^{\wedge k} \Upsilon^{\wedge}{}_{lq} u^{\wedge}{}_p \frac{\partial \Lambda_m}{\partial \nabla^{\wedge}{}_j E^{\wedge}{}_{pq}} - \Upsilon^{\circ}{}_{lq} \frac{\partial \Lambda_m}{\partial \nabla^{\wedge}{}_j E^{\wedge}{}_{qk}} + \frac{1}{2} u^{\wedge}{}_l \Upsilon^{\wedge}{}_p{}^j u^{\wedge}{}_q \frac{\partial \Lambda_m}{\partial \nabla^{\wedge}{}_k E^{\wedge}{}_{pq}} - \\ - \frac{1}{2} g^{\wedge js} g^{\wedge}{}_{lp} \Upsilon^{\circ}{}_{sq} \frac{\partial \Lambda_m}{\partial \nabla^{\wedge}{}_k E^{\wedge}{}_{pq}}$$

4. A generalisation of the model of an ideal compressible fluid. As another example, let us consider within the framework of the Special Theory some generalisation of the model of an ideal compressible fluid for the case when $\delta W^* = 0$. Let the defining parameters include the quantities

$$g^{\wedge pq}, \quad u^{\wedge p}, \quad \rho, \quad \partial \rho / \partial \xi^k$$

The corresponding model of Newtonian mechanics was considered by Eglit [6] for the case where the defining parameters included the derivatives of density ρ with respect to coordinates, and by Kogarko [7] for the case where the derivatives of ρ with respect to time were involved.

Performing the operations analogous to those carried out in the previous example, we obtain the following expressions for the canonical tensor of the energy impulse $P^{\wedge ij}$ and the tensor of 'double forces' $P^{\wedge ijk}$ in the associated reference system:

$$P^{\wedge ij} = 2g^{\wedge ip}g^{\wedge jq}\frac{\partial\Lambda}{\partial g^{\wedge pq}} + \left(u^{\wedge p}\frac{\partial\Lambda}{\partial u^{\wedge p}}\right)u^{\wedge i}u^{\wedge j} + \rho\frac{\partial\Lambda}{\partial\rho}\gamma^{\wedge ij} +$$
$$+ \frac{\partial\Lambda}{\partial(\partial\rho/\partial\xi^k)}\nabla^{\wedge}{}_k(\rho\gamma^{\wedge ij}) - \nabla^{\wedge}{}_k P^{\wedge ijk} - \Lambda g^{\wedge ij}$$
$$P^{\wedge ijk} = \frac{1}{2}\rho\left(\gamma^{\wedge ij}\frac{\partial\Lambda}{\partial(\partial\rho/\partial\xi^k)} + \gamma^{\wedge ik}\frac{\partial\Lambda}{\partial(\partial\rho/\partial\xi^j)}\right)$$

Tensor $M^{\wedge ijk}$, describing the internal moment of momentum and internal surface moments, is in the present case, usually different from zero. Indeed, in accordance with the definition of M^{ijk}, we have

$$M^{\wedge ijk} = P^{\wedge jik} - P^{\wedge ijk} = \frac{1}{2}\rho\left(\gamma^{\wedge jk}\frac{\partial\Lambda}{\partial(\partial\rho/\partial\xi^i)} - \gamma^{\wedge ik}\frac{\partial\Lambda}{\partial(\partial\rho/\partial\xi^j)}\right)$$

For $M^{\wedge ijk}$ from Equation (4.1), we have

$$\nabla^{\wedge}{}_k M^{\wedge ijk} = P^{\wedge ij} - P^{\wedge ji}$$

These relationships are identical with those in (1.14) derived on different assumptions.

In the case of an ideal compressible fluid, when Λ depends only upon the density ρ, the tensor of energy impulse $P^{\wedge ij}$ assumes the form

$$P^{\wedge ij} = \left(\rho\frac{\Lambda}{\partial\rho} - \Lambda\right)g^{\wedge ij} - \rho\frac{\partial\Lambda}{\partial\rho}u^{\wedge i}u^{\wedge j} = \rho^2\frac{\partial(\Lambda/\rho)}{\partial\rho}g^{\wedge ij} - \rho\frac{\partial\Lambda}{\partial\rho}u^{\wedge i}u^{\wedge j} =$$
$$= \rho^2\frac{\partial(\Lambda/\rho)}{\partial\rho}\gamma^{\wedge ij} - \Lambda u^{\wedge i}u^{\wedge j}$$

In this case the tensor of the energy impulse is already symmetric.

Let us now write in the proper reference system the components of the tensor of energy impulse for an ideal fluid, determined by means of the variational principle, together with the expressions which were obtained for those components on the basis of the relativistic mechanics of continuous media [3]:

$$\| P^{*ij} \| = \left\| \begin{matrix} -\rho^2\frac{\partial(\Lambda/\rho)}{\partial\rho} & 0 & 0 & 0 \\ 0 & -\rho^2\frac{\partial(\Lambda/\rho)}{\partial\rho} & 0 & 0 \\ 0 & 0 & -\rho^2\frac{\partial/(\Lambda/\rho)}{\partial\rho} & 0 \\ 0 & 0 & 0 & -\Lambda \end{matrix} \right\|, \qquad \| P^{*ij} \| = \left\| \begin{matrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & \rho U \end{matrix} \right\| \tag{4.2}$$

where U is the internal energy per unit mass, as measured by the observer in the proper reference system of each element of the fluid; p is the pressure, where in accordance with the first law of thermodynamics $p = \rho^2\partial U / \partial\rho$. Equating the matrices of (4.2) leads us to conclusion*, that

$$\Lambda = -\rho U$$

Let us now consider the properties of the energy impulse tensor in the particular case of an ideal compressible fluid whose internal energy is a function of the derivative of density ρ with respect to time.

$$\frac{d\rho}{ds} = u^{\wedge k}\nabla^{\wedge}{}_k\rho = u^{\wedge k}\frac{\partial\rho}{\partial\xi^k}$$

The individual study of such a model is of interest because, as shown by Kogarko [7] in Newtonian mechanics that, the model of cavitating fluid, as it happens, corresponds to the dependence of the internal energy on the time derivative of the density.

Let us assume that

$$\Lambda = -\rho(\Lambda_1 + \Lambda_2),\ \ \Lambda_1 = \Lambda_1(\rho),\ \ \Lambda_2 = \frac{1}{2}\lambda\left(\frac{d\rho}{ds}\right)^2 = \frac{1}{2}\lambda\,(u^{\wedge p}\nabla^{\wedge}{}_p\rho)\,(u^{\wedge q}\nabla^{\wedge}{}_q\,\rho)$$

where λ is a scalar not subject to variation. From (4.1) we obtain**

$$P^{\wedge ij} = -\left(p_1 + \rho\Lambda_2 - \frac{1}{2}\lambda\rho^2\frac{d^2\rho}{ds^2}\right)\gamma^{\wedge ij} + \left(\rho\Lambda_1 - \frac{d}{ds}\left(\frac{1}{2}\lambda\rho^2\frac{d\rho}{ds}\right)\right)u^{\wedge i}u^{\wedge j} + \\ + \nabla^{\wedge i}\left(\frac{1}{2}\lambda\rho^2\frac{d\rho}{ds}u^{\wedge j}\right) \qquad (4.3)$$

Here $p_1 = \rho^2\partial\Lambda_1 / \partial\rho$. The tensor of energy impulse is no longer symmetric, and it becomes necessary to take into account also the internal angular momentum of the fluid, connected with the tensor

$$M^{\wedge ijk} = P^{\wedge jik} - P^{\wedge ijk} = -\frac{1}{2}\lambda\rho^2\frac{d\rho}{ds}(\gamma^{\wedge ki}u^{\wedge j} - \gamma^{\wedge kj}u^{\wedge i}) = \\ = -\frac{1}{2}\lambda\rho^2\frac{d\rho}{ds}(g^{\wedge ki}u^{\wedge j} - g^{\wedge kj}u^{\wedge i})$$

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* In a metric with the sign convention (+++−) $\Lambda = \rho U$. This result was reached by Schopf [4] who used the variational principle in a different form and differently defined variations.

** Here we have also used the continuity equation $\nabla^{\wedge}{}_k(\rho u^{\wedge k}) = 0$. Indeed, taking into account the relationship $\det\|g^{\wedge}{}_{ij}\| = g^{\wedge} = \gamma^{\wedge} g^{\wedge}{}_{44}$, it is easy to verify that density ρ, as given by (3.3) satisfies the principle of conservation of mass [4]

$$\nabla^{\wedge}{}_k(\rho u^{\wedge k}) = \frac{1}{\sqrt{-g^{\wedge}}}\frac{\partial}{\partial\xi^k}\left(\sqrt{-g^{\wedge}}\,u^{\wedge k}\right) = \\ = \frac{1}{\sqrt{-g^{\wedge}}}\frac{\partial}{\partial\xi^k}\left(\sqrt{-g^{\wedge}}\frac{\sqrt{-\gamma^0}}{\sqrt{-g^{\wedge}}}\times\sqrt{g^{\wedge}{}_{44}}\frac{\delta_4{}^k}{\sqrt{g^{\wedge}{}_{44}}}\right) = \frac{1}{\sqrt{-g^{\wedge}}}\frac{\partial}{\partial\xi^4}\sqrt{-\gamma^{\circ}} = 0$$

Here also

$$M^{\wedge\alpha\beta k}=0, \qquad M^{\wedge\alpha 4k}=-M^{\wedge 4\alpha k}=-\frac{1}{2}\lambda\rho^2\frac{d\rho}{ds}g^{\wedge\alpha k}$$

But

$$\nabla^{\wedge}{}_k M^{\wedge\alpha\beta k}\neq 0, \quad \nabla^{\wedge}{}_k M^{\wedge\alpha 4k}\neq-\frac{1}{2}\nabla^{\wedge}{}_k\left(\lambda\rho^2\frac{d\rho}{ds}g^{\wedge\alpha k}\right)$$

The stress tensor is also no longer symmetric, and in the proper coordinate system we have for it, from (4.3),

$$p^{*\alpha\beta}=\left(p_1+\rho\Lambda_2-\frac{1}{2}\lambda\rho^2\frac{d^2\rho}{ds^2}\right)g^{*\alpha\beta}-\frac{\partial}{\partial x_\alpha{}^*}\left(\frac{1}{2}\lambda\rho^2\frac{d\rho}{ds}u^{*\beta}\right)$$

$$g^{*11}=g^{*22}=g^{*33}=-1, \qquad g^{*\alpha\beta}=0 \text{ when } \alpha\neq\beta$$

Here, just asin the Newtonian mechanics, the stress tensor is a linear function of the second time derivatives of ρ but, unlike in Kogarko's model, the stress tensor is not spherical.

Let us write down the expressions for the following components of the energy impulse tensor

$$P^*{}_4{}^\alpha=\frac{\partial}{\partial x^{*4}}\left(\frac{1}{2}\lambda\rho^2\frac{d\rho}{ds}u^{*\alpha}\right), \qquad P^*{}_\alpha{}^4=\frac{\partial}{\partial x^{*\alpha}}\left(\frac{1}{2}\lambda\rho^2\frac{d\rho}{ds}\right), \qquad P^*{}_4{}^4=\rho\Lambda_1 \quad (4.4)$$

This equation shows that, in the present model of the fluid, each particle experiences an adiabatic inflow of energy which is described by a three-dimensional vector $P^*{}_4{}^\alpha$. This inflow of energy depends not only on the first and second time derivatives of the density of the particle, but also of its acceleration. In addition, in the associated reference system, there is a macroscopic transfer of momentum which is defined by a three-dimensional vector $P^*{}_\alpha{}^4$. Equation (4.4) also shows that the components of vector $P^*{}_4{}^4$ are not identical to $(-\Lambda)$.

Internal energy U, in general also contains the entropy S as one of its arguments. However, if S were included in the arguments of Λ, one would obtain

$$\frac{\partial\Lambda}{\partial S}=0$$

For the model of an ideal fluid one would thus have

$$\frac{\partial U}{\partial S}=0$$

The latter is a result of the assumption that $\delta W^*=0$. Since, for an arbitrary process $\partial U\,/\,\partial S\neq 0$, it is necessary to assume that δW^* includes the integral

$$\int\limits_V Q_A\delta\mu^A\,d\tau$$

where for $\mu^A=S$ the corresponding component of Q_A per unit mass has the dimension of absolute temperature T, and the integral appearing in δW^* is of the form

$$\int\limits_V \rho T\delta S\,d\tau$$

As the Lagrangian of an ideal fluid or an elastic body one can take not only the internal energy, but also any other thermodynamic potential, such as, for instance, the density of the free energy ρF. In this case, the expression for the stress tensor will remain the same, but the component $P^{\wedge}{}_4{}^4$ will have a different meaning, namely $P^{\wedge}{}_4{}^4 = \rho F$. Since the arguments of the free energy F include the absolute temperature, the integral over the region V, appearing in δW^*, will be in the form

$$-\int_V \rho S \delta T \, d\tau$$

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